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ON THE THEORY OF INEXTENSIONAL BENDING OF SHELL STRUCTURES

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Abstract—The theory of inextensional bending of the middle surface of a shell is closely connected with the geometrical problem of the bending of surfaces. Still it is difficult to apply the known solutions of these problems of differential geometry to actual shell structures. For their sake and safety, however, the knowledge of the shell specific inextensional bending has proved to be useful in many engineering applications. It is indispensable in the analysis of shells where the boundary conditions are given in terms of deformations.

This paper reviews the present state of inextensional bending and presents the geometrical and the physical problem as well. Main emphasis has been given to various forms of mathematical treatment offering a new approach. Numerical results illustrate the procedure and provide a better understanding of the problem of inextensional bending of shell structures.

1. INTRODUCTION

Proceeding from the global field equations valid for the entire sphere of continuum mechanics, the behaviour of any body can be described by means of certain constitutive equations. A particularly refined form of this procedure is used for the derivation of shell theories which are nowadays more or less free from contradictions and have been published in numerous recent papers. If, therefore, the derivation of the "best" theory for shell structures can be regarded as being well established, in practical application the direct numerical solution of arbitrarily complex structures proves to be either too laborious or non solvable with present-day mathematical means. For this reason it is to be recommended that fundamental simplifications be made by defining special cases of the shell theory (see, e.g. [8, 11, 13, 14, 20]).

A graph of possible special cases of practical significance is shown in Fig. 1. If one classifies the special cases by means of the curvature $1/R$ plotted on the axis of ordinates, then all the plane problems are obtained on the axis of abscissas with the extreme cases of pure plate bending and of plane stress, respectively. Quite analogously one obtains for $1/R \neq 0$ the general bending theory with the extreme special cases of the membrane theory and of the inextensional bending of the middle surface of the shell. This latter special case is one of the least-investigated

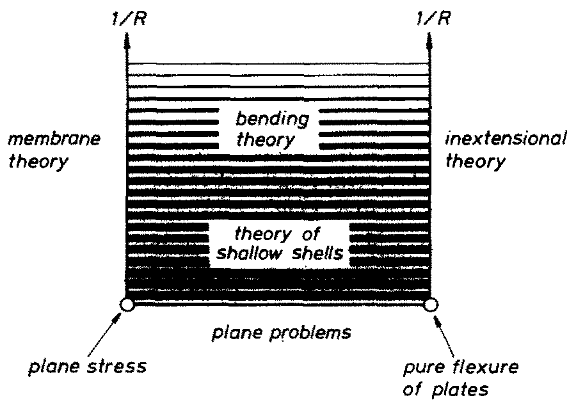

Fig. 1. Special cases of shell theory.

phenomena in shell theory. Apart from the publications of the authors[9, 19], special mention is due to those of Duddeck[3], Fluegge[5], Fluegge–Geyling[6], and Green–Zerna[8]. The following chapters shall give an idea of the present state of the theory of inextensional bending within the scope of the shell theory.

To justify the classification according to geometrical aspects as chosen here it has to be pointed out that the geometry of spatially curved structures is of fundamental importance. In contrast to plates and in-plane panels, shells will carry their loadings in principle not because of the strength of their material but rather because of their geometrical properties. If optimally chosen, shells combine a maximum of load-carrying capacity and stability of form with a minimum of the amount of the material used. Examples of such structures are known to us in nature in the shells of birds' eggs and in the structure of cells. Dealing with actual engineering problems such optimum structures can usually not be realized since it is necessary to have openings of various kind and arrangement for functional reasons. Due to this "interference" with the "natural" condition, the stiffening of the artificially provided boundaries becomes a matter of prime importance in that the load-carrying properties of the removed parts of the shell must be compensated, so that the particularly undesirable accompaniments of inextensional bending are avoided.

On behalf of the small transverse dimensions of shell structures in relation to their longitudinal dimensions it seems to be advantageous to represent the inextensional bending of a shell as a problem of differential geometry of its middle surface (see Section 2). The question of satisfactory boundary conditions can then be answered reasonably from the available mathematical literature[4]. In many a problem of practical engineering, however, the dimension of finite thickness of the real structure unfortunately cannot be neglected, not even as an initial approximation. In such cases the bending stiffness of the spatial structure offers considerable resistance to its being deformed. This becomes apparent in secondary membrane forces, so that in real structures generally there does not exist a case of exact inextensional bending. How this complex reality in regard to shell structures can be dealt with by means of a mechanical-mathematical model susceptible to computation will be explained in Sections 3 and 4. This has been achieved here by means of an infinitesimal shell theory and can be extended analogously to a theory of large deformations. In the section following thereafter, the main intention is to find possible solutions for analytical methods considering "ideal" surfaces and for numerical investigations aimed at attaining the "complete" theory. Experimental investigations are only considered marginally. In conclusion, characteristic examples of numerical results will be given to illustrate the new approach presented in this paper.

To increase the scope of present investigations by including large deformations, the consideration of quasi-inextensional bending may also be employed in setting up a stability theory of shells[7].

The representation of the theory is based on the tensor calculus introducing the notation of Ricci, and agrees basically with [8]. Use is made of the Einstein summation convention, according to which summation takes place over opposite indices of the same kind. The Greek letters take the values of 1 and 2. Vectors are indicated by bold letters. On the surface the Gaussian parameters θ^α are used. The partial derivation according to these coordinates is represented by (,), the covariant derivation by (|). The scalar product of two vectors is indicated by (·), the vectorial product by (x).

2. THE GEOMETRICAL PROBLEM

The aim of any physical theory is to describe phenomena and processes of the empirical world surrounding us. The complete ascertainment of all influences is bound to fail at the present time already because of mathematical difficulties. One is thus compelled to a series of simplifications and neglections. This sometimes takes place in a very radical way in order to first investigate the most important characteristics and only afterwards, bit by bit, to ascertain parts of the neglected influences and their effects. With this in mind, as a first step towards defining and investigating inextensional bending, the "real" shell structure is therefore abstracted into an "ideal" surface. This method seems reasonable and suitable, since the stress and deformation behaviour of shell structures, too, is generally described by that of a reference surface resistant to bending and stretching.

As a result of this idealization, a mathematically consistent formulation of the geometrical problem is now possible. This is based upon the differential geometry of surfaces, in which particularly their deformation plays an important part. A relatively detailed representation has been given by Harnach in[9, 10], while more specific details are to be found, for instance, in[4, 15, 16].

The original, that is to say undeformed, surface is indicated by F, and the deformed surface by f. On F the Gaussian parameters θ^α are chosen as curvilinear convective coordinates. If one further assumes that there exists a permissible projection between F and f, then f can also be related to the same parameters θ^α [9, 15]. Thus a completely analogous description of both surfaces is possible. Any point of F resp. f can be described by the position vectors of the undeformed resp. deformed surface as follows:

$$\mathbf{R} = \mathbf{R}(\theta^\alpha) \quad \text{resp.} \quad \mathbf{r} = \mathbf{r}(\theta^\alpha). \tag{2.1}$$

On F resp. f a vector basis can be defined by

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} \quad \text{resp.} \quad \mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} = \mathbf{r}_{,\alpha}. \tag{2.2}$$

If one further assumes for the vectorical products

$$\mathbf{A}_1 \times \mathbf{A}_2 \neq 0 \quad \text{resp.} \quad \mathbf{a}_1 \times \mathbf{a}_2 \neq 0 \tag{2.3}$$

then the unit normal vectors of F resp. f become

$$\mathbf{A} = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} \quad \text{resp.} \quad \mathbf{a} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}. \tag{2.4}$$

The three vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}$ (resp. $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}$) form a non orthonormal moving trihedral on F (resp. f). This should be oriented in such a way that the vectors in the given sequence form a right-handed system. The first and second fundamental forms of F (resp. f) are

$$d\mathbf{R}\cdot d\mathbf{R} = A_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta \quad \text{resp.} \quad d\mathbf{r}\cdot d\mathbf{r} = a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta, \tag{2.5}$$

$$d\mathbf{R}\cdot d\mathbf{A} = -B_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta \quad \text{resp.} \quad d\mathbf{r}\cdot d\mathbf{a} = -b_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta. \tag{2.6}$$

The symmetric covariant metric tensor is characterized by

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta \quad \text{resp.} \quad a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \tag{2.7}$$

and the symmetric covariant curvature tensor by

$$B_{\alpha\beta} = -\mathbf{A}_\alpha \cdot \mathbf{A}_{,\beta} \quad \text{resp.} \quad b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{,\beta}. \tag{2.8}$$

Both tensors are of basic importance, which results from the fundamental theorem of the theory of surfaces. According to this theorem, the surface F (resp. f) is well defined by $A_{\alpha\beta}$ and $B_{\alpha\beta}$ (resp. $a_{\alpha\beta}$ and $b_{\alpha\beta}$) except for rigid body motions. The vectors of the moving trihedral must, in this case, fulfil integrability conditions, respectively the fundamental tensors must satisfy the well-known equations of Gauss and Mainardi–Codazzi.

For the difference of the two fundamental forms of F and f one obtains

$$d\mathbf{r}\cdot d\mathbf{r} - d\mathbf{R}\cdot d\mathbf{R} = 2\alpha_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta, \tag{2.9}$$

$$d\mathbf{r}\cdot d\mathbf{a} - d\mathbf{R}\cdot d\mathbf{A} = \omega_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta. \tag{2.10}$$

The difference tensors

$$2\alpha_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{2.11}$$

$$\omega_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} \tag{2.12}$$

describe, according to definition, the metric and curvature changes because of the transformation from F to f, that is to say, because of the surface deformation. The relation between the surfaces F and f, resp. between their position vectors (2.1), is now established by equation, see also Fig. 2,

$$\mathbf{r}(\theta^\alpha) = \mathbf{R}(\theta^\alpha) + \mathbf{v}(\theta^\alpha), \tag{2.13}$$

in which $\mathbf{v}$ represents the deformation vector of F. If this is not subjected to any limitations to its order of magnitude, then one speaks of a finite deformation of the surface F into the surface f. With (2.13) one obtains for the moving trihedral of f

$$\mathbf{a}_\alpha = \mathbf{A}_\alpha + \mathbf{v}_{,\alpha}, \quad \mathbf{a} = \mathbf{A} + \mathbf{w}, \tag{2.14}$$

in which case the normal change vector $\mathbf{w}$ over $(2.4)_1$ and $(2.14)_1$ can be represented by $\mathbf{v}$. For (2.11) and (2.12) one can further obtain

$$2\alpha_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{v}_{,\beta} + \mathbf{A}_\beta \cdot \mathbf{v}_{,\alpha} + \mathbf{v}_{,\alpha} \cdot \mathbf{v}_{,\beta}, \tag{2.15}$$

$$\omega_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{w}_{,\beta} + \mathbf{A}_{,\beta} \cdot \mathbf{v}_{,\alpha} + \mathbf{v}_{,\alpha} \cdot \mathbf{w}_{,\beta}. \tag{2.16}$$

The two difference tensors $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are of fundamental significance for the deformation of a surface. This follows directly from (2.11) and (2.12) and from the fundamental theorem of the theory of surfaces. Viewed from this standpoint, the use of other deformation quantities seems less reasonable (see [9, 10]).

If one now turns to the global theory of surfaces and presupposes the surfaces to be limited and orientable, then a further important uniqueness theorem can be obtained. According to this theorem, the two surfaces F and f are congruent if their fundamental tensors agree at all points (again with the exception of a rigid body motion). The condition of congruence of the surfaces F and f then follows directly from (2.11) and (2.12) by

$$\alpha_{\alpha\beta} = 0 \quad \text{and} \quad \omega_{\alpha\beta} = 0. \tag{2.17}$$

Since, however, in the case of a general deformation (2.13), $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$ will usually not disappear simultaneously, it is possible to define the further special case

$$\alpha_{\alpha\beta} = 0 \quad \text{and} \quad \omega_{\alpha\beta} \neq 0 \tag{2.18}$$

which, in particular, describes the bending of a surface. A finite deformation (2.13) is denoted as a finite bending if, during the process of deformation, all the arc lengths remain unaltered. If one denotes the line elements of f and F with $\mathrm{d}s$ and $\mathrm{d}S$ respectively, then (2.5) represents their squares, and, because of (2.9) and (2.11), the equations

$$\mathrm{d}s^2 = \mathrm{d}S^2 \quad \text{resp.} \quad a_{\alpha\beta} = A_{\alpha\beta} \tag{2.19}$$

must apply to all points of the surface. With (2.15) one obtains from (2.18) the equivalent

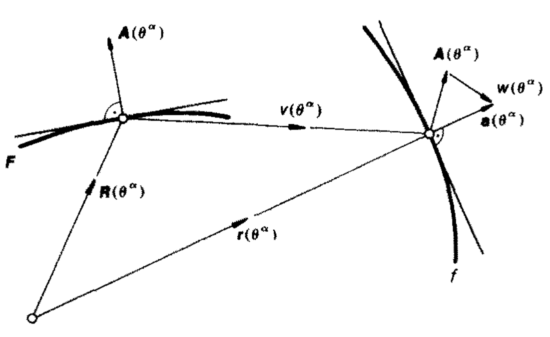

Fig. 2. Description of the undeformed and the deformed middle surface of the shell.

statement

$$\alpha_{\alpha\beta}=0 \quad \text{resp.} \quad \frac{1}{2}(\mathbf{A}_\alpha\cdot\mathbf{v}_{,\beta}+\mathbf{A}_\beta\cdot\mathbf{v}_{,\alpha}+\mathbf{v}_{,\alpha}\cdot\mathbf{v}_{,\beta})=0. \tag{2.20}$$

The eqns (2.19) and (2.20) represent equivalent equations of conditions for the existence of a surface bending. If, in an exceptional case, (2.17) is also satisfied, then the bending of the surface degenerates into a rigid body motion.

In substance a surface-bending represents a length-preserving (isometric) projection, which naturally at the same time is also angle-preserving (conformal) and area-preserving. The equation (2.19) also states that all quantities related to the first fundamental form remain unchanged, that is to say, are bending-invariant.

Regarding practical engineering problems one is often able to classify an infinitesimal deformation within the scope of a geometrically linear theory. The finite deformation (2.13) is then qualified in that sense that products of $\mathbf{v}$ as against $\mathbf{v}$ itself may be neglected. This must also apply to the derivatives of $\mathbf{v}$. The simplifications

$$2\alpha_{\alpha\beta}=\mathbf{A}_\alpha\cdot\mathbf{v}_{,\beta}+\mathbf{A}_{,\beta}\cdot\mathbf{v}_{,\alpha}, \tag{2.21}$$

$$\omega_{\alpha\beta}=\mathbf{A}_\alpha\cdot\mathbf{w}_{,\beta}+\mathbf{A}_{,\beta}\cdot\mathbf{v}_{,\alpha} \tag{2.22}$$

are then obtained for (2.15) and (2.16), i.e. quantities of higher order (such as $\mathbf{v}_{,\alpha}\cdot\mathbf{v}_{,\beta}$) may be neglected in regard to $\mathbf{v}$. An infinitesimal deformation is now denoted as an infinitesimal bending if all arc lengths with the exception of quantities of second (and higher) order(s) remain unchanged. From a comparison of the line elements of f and F the equation of condition

$$d\mathbf{R}\cdot d\mathbf{v}=0 \tag{2.23}$$

is obtained, since $d\mathbf{v}\cdot d\mathbf{v}$ as a quantity of the second order has been neglected. Taking into consideration the already simplified form (2.21) one obtains in analogy with (2.20)

$$\alpha_{\alpha\beta}=0 \quad \text{resp.} \quad \frac{1}{2}(\mathbf{A}_\alpha\cdot\mathbf{v}_{,\beta}+\mathbf{A}_\beta\cdot\mathbf{v}_{,\alpha})=0. \tag{2.24}$$

While $(2.24)_1$ and $(2.20)_1$ formally agree, a comparison of $(2.24)_2$ with $(2.20)_2$ clearly shows the difference between a finite and an infinitesimal bending. Indeed, one may no longer speak of a length-preserving projection, since in the case of an infinitesimal bending a factual change of length is permissible, even if it is of a smallness of higher order. One thus recognizes that the definition of infinitesimal bending is related to a much weaker geometrical condition than is that of finite bending. In [4] it has been shown that one can very well define bending theories of a higher order—a possibility into which it is not intended, however, to go in more detail here.

A simple example of a surface bending can be demonstrated with a sheet of paper, which indeed comes very close to the mathematical model of a surface. This sheet of paper can be given various curved configurations without tearing it, i.e. without having any change of length. Corresponding deformations are also possible at a cylindrical model, which can be formed by sticking together two opposite edges of a sheet of paper. The fact that there are also surfaces at which such deflections are not possible will be discussed in Section 5. In this case the surfaces are denoted as nonbendable. In order to emphasize more clearly the difference between a finite and an infinitesimal theory the denotations nonrigidity and rigidity for infinitesimal bendability resp. nonbendability are commonly used. Taking into account the mutual dependencies we get the following relations

finite theory:	bendable	nonbendable
	↓	↑
infinitesimal theory:	nonrigid	rigid

The arrows indicate the direction of the deduction, i.e. a rigid surface is always nonbendable and a bendable surface is always nonrigid. It should be noted, however, that the reverse deduction is not always possible (see [4, 9]).

In the sense of differential geometry the problem of the bending of surfaces was defined as a deformation without (resp. only sufficiently small) changes in length. In the case of applying the two difference tensors this is equivalent to the fact that during bending there are no (resp. only sufficiently small) metric changes, but only curvature changes. The determination of the bendability or nonrigidity of a surface would now mean to determine a compatible deformation vector v in such a way that the condition $\alpha_{\alpha\beta} = 0$ is fulfilled at any point and only changes of curvature $\omega_{\alpha\beta} \neq 0$ occur. If such a vector v cannot be found, i.e. if $\omega_{\alpha\beta} = 0$ would always apply, then the surface according to (2.17) is nonbendable resp. rigid. Several methods for such investigations will be given in Section 5. Then it will become evident that particularly the type of curvature and the boundary conditions of the surface are of decisive importance.

The two previously-mentioned examples illustrating the bending (with the aid of a sheet of paper) show very clearly that bendable surfaces possess a high degree of deformability, i.e. they are very soft and flexible. Thus, generally speaking, they are completely useless for technical shell structures of concrete or steel. If, however, one considers the idealization introduced at the beginning of this section, one recognizes that the results gained from a model surface have only limited validity for the real structure. Actually existing spatial structures are, in fact, not surfaces in a strictly mathematical sense. Rather do they possess an extension—however small—in the third direction. It is natural, however, that the influence of the thickness of a shell structure cannot be determined by a theory of surfaces. To explain this fact let us look at a shell with a constant thickness t. In accordance with (2.9) the difference between the squares of the line elements on the deformed and undeformed middle surface of the shell

$$ds^2 - dS^2 = 2\alpha_{\alpha\beta} d\theta^\alpha d\theta^\beta \tag{2.25}$$

describes the change of length in this surface, caused by the deformation. If one considers a surface parallel to the middle surface of the shell at a distance of z $(-t/2 \le z \le +t/2)$, then one obtains for infinitesimal deformations

$$ds^2 - dS^2 = (2\alpha_{\alpha\beta} + 2z\omega_{\alpha\beta} - z^2[B_\alpha{}^\lambda \omega_{\lambda\beta} + B_\beta{}^\lambda \omega_{\lambda\alpha} + 2B_\beta{}^\rho b_\alpha{}^\lambda \alpha_{\rho\lambda}])d\theta^\alpha\, d\theta^\beta, \tag{2.26}$$

$B_\alpha{}^\beta$ being the mixed variant curvature tensor. If in accordance with the definition for the bending of the middle surface of the shell one equates $\alpha_{\alpha\beta}$ to zero, then the terms in (2.26) affected by $\omega_{\alpha\beta}$ will remain. This means, however, that in the case of a shell (with a thickness t), changes in length actually do occur, although the middle surface itself does not suffer any changes in length. As a result of this effect the so-called "bending stiffness" of the shell is brought into play, which sets up a resistance against the bending of its middle surface. It can now be recognized from this effect, however, that the thickness of the shell can be of importance, since the degree of bendability of the middle surface of the shell is influenced by its bending stiffness. From this it can be concluded that a "bendable" shell will, on the whole, possess less deformability than might be expected on behalf of the bendability of its middle surface.

To which extent the bending stiffness may become of decisive importance, can be uniquely demonstrated at plane structures, i.e. at plates. In the Kirchhoff theory of plates it is usually accepted that one may neglect the elongation and angular changes of the middle surface of the plate. Only deformations out of the plane—that is to say, bendings—are considered. Since the deformations are assumed to be sufficiently small, such "inextensional" deformations, due to the nonrigidity of a circumferentially-supported plane surface [9], are actually possible. Thus the Kirchhoff theory of plates represents the classical case of a theory of bending. The remarkable thing about this is that the load-carrying behaviour of a plate is mainly determined by its bending stiffness (in addition to the nature of the boundary conditions). Since, however, a circumferentially-supported plane surface is always nonbendable [9], a theory of pure bending does not suffice in the case of finite deformations; i.e. besides the moments, the normal forces must also be considered.

3. THE PHYSICAL PROBLEM

What we have considered up to now has been mainly of a geometrical nature and was related to an "ideal" surface. Regarding real shell structures, additional influences appear which also have to be taken into consideration. One of these is the structural thickness (resp. the bending stiffness), the effect of which has been discussed in the previous section. Another very important factor are the properties of the building material. It is well known that the strain measures are related to the resulting stresses and stress couples by means of constitutive equations, and the stresses or stress resultants are of main interest in structural computation. The development of suitable constitutive equations presents quite a problem in itself and requires further investigation. This whole question will be explained here considering a geometrical and physical linear shell theory.

Proceeding from the three-dimensional continuum, the constitutive equations of the linear shell theory can be derived with the aid of various simplifications and assumptions within the Boltzmann theorem of mechanics. Generally speaking, this takes place by means of considerations of order of magnitudes. In particular the shell parameter

$$\lambda = t/L \ll 1 \tag{3.1}$$

plays an important part, L being a characteristic length of the middle surface of the shell. By consistently not taking into account those terms affected by λ^2[11, 13, 14], a first approximation (after introducing the hypothesis of Love–Kirchhoff) produces the following coordinations

$$\tilde{n}^{\alpha\beta} = \tilde{n}^{\alpha\beta}(\alpha_{\alpha\beta}) = DH^{\alpha\beta\rho\lambda}\alpha_{\rho\lambda}, \quad \tilde{n}^{\alpha\beta} = \tilde{n}^{\beta\alpha}, \tag{3.2}$$

$$m^{\alpha\beta} = m^{\alpha\beta}(\omega_{\alpha\beta}) = BH^{\alpha\beta\rho\lambda}\omega_{\rho\lambda}, \; m^{\alpha\beta} = m^{\beta\alpha}. \tag{3.3}$$

In these

$$\tilde{n}^{\alpha\beta} = \int_{-t/2}^{+t/2} h\tau^{\alpha\beta}\,\mathrm{d}z \quad \text{and} \quad m^{\alpha\beta} = \int_{-t/2}^{+t/2} h\tau^{\alpha\beta}z\,\mathrm{d}z \tag{3.4}$$

represent the symmetrized membrane force tensor and the symmetrical moment tensor. The term h is defined by

$$h = 1 - zb_\lambda{}^\lambda + z^2[b_1{}^1b_2{}^2 - b_1{}^2b_2{}^1] \tag{3.5}$$

with z representing the coordinates in the third direction normal to the middle surface. The membrane stiffness and the bending stiffness of the shell are denoted by

$$D = \frac{Et}{1-\nu^2} \quad \text{and} \quad B = \frac{Et^3}{12(1-\nu^2)}. \tag{3.6}$$

The material properties used are Young's modulus of elasticity E and Poisson ratio ν. For the symmetrical tensor of elasticity the equation

$$H^{\alpha\beta\rho\lambda} = \frac{1}{2}[(1-\nu)(a^{\alpha\lambda}a^{\beta\rho} + a^{\alpha\rho}a^{\beta\lambda}) + 2\nu a^{\alpha\beta}a^{\rho\lambda}] \tag{3.7}$$

applies, $a^{\alpha\beta}$ being the contravariant metric tensor. Other forms of representation for the constitutive equations have been discussed by Harnach in[11], but it is not intended to go into further details about them here.

If in agreement with the definition of the bending of a surface one now equates $\alpha_{\alpha\beta}$ to zero, then in accordance with (3.2) $\tilde{n}^{\alpha\beta}$ will also disappear. From the simplified conditions of compatibility according to[10]

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\sigma}b_{\beta\gamma}\omega_{\alpha\sigma} = 0, \quad \epsilon^{\alpha\beta}\epsilon^{\omega\gamma}\omega_{\gamma\alpha|\beta} = 0 \tag{3.8}$$

one can calculate the difference tensor $\omega_{\alpha\beta}$ and subsequently from (3.3) also the moment tensor $m^{\alpha\beta}$. At this point one recognizes very clearly that in this special case of the bending theory of shells the state of stress of the shell is quite decisively influenced by the moments. A shell structure, in which such a state is possible, will, in the case of appropriate boundary conditions, be very soft and, because of its flexibility towards external loadings, should be avoided for practical purposes. From this, one may deduce that a shell will then also be very flexible (that is to say, will show severe bending stresses) when $\alpha_{\alpha\beta}$ on behalf of the given loading remains sufficiently small. In fact, the extreme case $\alpha_{\alpha\beta} = 0$ is not likely to occur very often. That means, however, that in reality there is a soft transition from the general bending theory of shells to the special case of inextensional bending of the middle surface. The reason for this is mainly that the formulation of shell theories—as mentioned at the beginning—is done with various consideration of the order of magnitudes of certain terms. Generally speaking, these only allow a very global and general estimate of borderline cases, which always contain an "area of uncertainty".

It is therefore obvious that the definition of inextensional bending cannot only be confined to the narrowly limited special case $\alpha_{\alpha\beta} = 0$, but must also permit of (relatively small) changes in length in the middle surface of the shell. With the aid of the shell parameter λ "inextensional" bending is generally accepted to exist when $\alpha_{\alpha\beta}$ in comparison to $\omega_{\alpha\beta}$ is at the most of the magnitude λ^2. In order to limit the "area of uncertainty" Basar and Rothert[2] introduced the new equation of condition

$$\max|\alpha_{\alpha\beta}| \leq \lambda^{2-r} \max|\omega_{\alpha\beta}| \quad \text{with} \quad 0 < r < 1. \tag{3.9}$$

This new definition does not, however, represent a departure from the existing, mathematically strict definition $(2.24)_1$. Rather is (3.9) embedded in the uncertainty area of shell theory and is only meant to provide a rough working basis for becoming aware of inextensional bending. This also corresponds more closely with the intent of the theory to investigate shell forms as to the condition under which they are soft and flexible.

Because of (3.9) one is also able to calculate membrane forces with the aid of (3.2). In connection with the initially-described approximation one must find out whether or not the constitutive equations (3.2) and (3.3) may require an extension of the form

$$\bar{n}^{\alpha\beta} = \bar{n}^{\alpha\beta}(\alpha_{\alpha\beta}, \omega_{\alpha\beta}), \quad m^{\alpha\beta} = m^{\alpha\beta}(\alpha_{\alpha\beta}, \omega_{\alpha\beta}). \tag{3.10}$$

For the general case of the bending theory of shells, where $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$ can be of comparable magnitude, this question has been investigated by Kraetzig[13, 14]. He describes a further area of uncertainty for the tensors of elasticity. Nevertheless, it seems that it cannot be regarded as out of the question that for (2.24) or (3.9) a higher approximation of the constitutive equations is possible and necessary.

Attention is drawn to a further peculiarity. The constitutive equations (3.2), (3.3) were determined on the assumption of the hypothesis of Love–Kirchhoff. If one discards this presupposition, then shearing deformations will also occur, and constitutive equations can then likewise be given for the shearing forces q^α (see[13]). In the case of a comprehensive investigation of inextensional bending the influence of shearing stiffness ought also to be considered.

Taking all the circumstances described into consideration, the constitutive equations appear to be the least certain part of the whole theory up to now, particularly as the field of finite deformations with its much more complicated relations has not yet been studied at all.

4. THE STATICAL PROBLEM

After having chosen the difference tensors $\alpha_{\alpha\beta}$ and $\omega_{\alpha\beta}$ as geometrical variables (strain measures), the static variables (stress resultants) used in the theory were determined by means of the constitutive equations (3.2), (3.3). In[10] Harnach has described in detail how the geometric variables are related to one another over the compatibility conditions. Quite analogously, the static variables are also not independent of one another. With inclusion of the given external loads they have to be in equilibrium. These conditions of equilibrium are represented in the linear shell theory e.g. in the form

$$n^{\alpha\beta}|_\alpha - b_\lambda{}^\beta m^{\alpha\lambda}|_\alpha + p^\beta = 0, \tag{4.1}$$

$$b_{\alpha\beta} n^{\alpha\beta} + m^{\alpha\beta}|_{\alpha\beta} + p = 0. \tag{4.2}$$

The shearing forces q^β have already been eliminated by

$$q^\beta = m^{\alpha\beta}|_\alpha. \tag{4.3}$$

The tangential and normal components of the external load vector have been denoted by p^β and p, external load moments have not been taken into consideration. The connection between $\bar{n}^{\alpha\beta}$ and $n^{\alpha\beta}$ is given by the condition of symmetry

$$\bar{n}^{\alpha\beta} = n^{\alpha\beta} + b_\lambda{}^\beta m^{\lambda\alpha}. \tag{4.4}$$

A consistent derivation of the conditions of equilibrium can be taken, for example, from the laws of thermodynamics[14, 17], or the variational principles[11, 18]. With the latter method one obtains, at the same time, the pertinent boundary conditions. These have hitherto not received attention, although they play a very considerable part. It has already been pointed out in Section 2 that the rigidity or bendability of a surface is decisively influenced by the boundary conditions. This fact is implicitly already embodied in equation (3.8), since it is natural that in the integration of this system of equations the boundary conditions must be taken into account.

The influence of the boundary conditions can be illustrated by a simple example. In Section 5 it will be shown that in the sense of the bending of a surface a cylinder closed at the ends by a plane surface is rigid. If one cuts a hole into these plane surfaces (without touching the bounding curve of the cylinder), the cylinder still remains rigid. If one now turns again to a real structure, a number of additional aspects emerge. If one, at first, replaces the base of the cylindrical shell by a panel, then the shell, because of its elastic tensility, will no longer be completely rigid, but nevertheless one may suppose that there will be a certain degree of flexibility of the cylindrical shell. If one now replaces the perforated base surface by a circular ring beam, then it is nearly impossible to make any statement as to the flexibility of the shell in a general form since the stiffness of the circular ring beam plays a decisive part. In a borderline case it tends towards zero, and it would follow the nonrigidity or, as the case may be, a large degree of flexibility of the shell. This fact is also known from the theory of cylindrical roofs. Of great interest to building engineering in this respect would be an investigation as to what degree of stiffness a ring beam must have in order to considerably reduce the deformability of a shell.

The influence just described is based chiefly on the elastic flexibility of the edge support. Kollar[12] makes use of the expression "pseudorigid" and provides an explanation for the case of the circular cylindrical shell.

Deviating from this, there are cases in which the influence of the supporting mode is of minor importance. In this connection we shall take as an example a "long circular cylindrical shell" with end-panels. As a surface model, rigidity would be ensured; as a real structure, however, it is possible that—independent of the supporting mode—there may be large deformations in the middle of the span, particularly if the loads only act there and are subjected to considerable changes in the circumferential direction. Here again the bending stiffness of the shell plays an important part.

The remarks made up to now show that various influences exceeding the context of a surface model—such as wall thickness, material properties, loadings, and boundary conditions—occur and exercise different effects. The fact is, however, that not only the complete description of the problem of inextensional bending is of an extremely complex nature but also its solution. Dependent on the practical requirements in each case, one will need to look for suitable methods for the solution of the problems in order to achieve, with as little effort as possible, a sufficiently accurate result. In the following section a number of approaches will be described.

5. POSSIBILITIES FOR SOLUTIONS

At first one can—as described in Section 2—again proceed from the abstract model of an ideal surface and investigate its deformation behaviour, that is to say, make use of methods of differential geometry. This has been dealt with in great detail from the mathematical point of view in[4]. A presentation of the subject matter in general, adequate for engineering requirements, can be found in[9, 10]. In the following remarks, therefore, only the most important facts are summarized. The definition of a finite or an infinitesimal bending of a surface is given in (2.20) and

(2.24) respectively. Both equations may be traced back to partial differential equations—(2.20) or, as the case may be (2.24)—for the deformation vector v. If the only solution is the trivial case of $\mathbf{v} = 0$, then, because of (2.13), $\mathbf{r} = \mathbf{R}$ would always hold good, i.e. both surfaces agree identically. The other trivial case $\mathbf{v} = \text{const.}$ describes a pure rigid body translation. In both cases a congruity of the surfaces then exists. Accordingly, every nontrivial solution of the differential equation would describe a surface-bending. In general, however, the integration of partial differential equations is rather difficult, as can be gathered from pertinent literature. It is therefore not intended to attempt a more detailed discussion of the determination of surface-bendings here. For problems of practical engineering, the question is, in fact, most often one of finding out the rigidity or nonbendability of a surface, since in such cases a greater flexibility of the shell structure is normally not to be expected. Proceeding from the fundamental theorem of the theory of surfaces and the uniqueness theorem in a global consideration, it could thus be proved that at all points of the surface the conditions

$$\omega_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} = 0 \quad \text{and} \quad 2\alpha_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta} = 0 \tag{5.1}$$

are simultaneously fulfilled. Since a surface-bending is defined by the condition $\alpha_{\alpha\beta} = 0$, all variables of the bent surface will, for purposes of brevity of the representation, in future be indicated by an (*), so that (5.1) can be replaced by

$$\omega_{\alpha\beta} = B_{\alpha\beta} - b^*_{\alpha\beta} = 0. \tag{5.2}$$

Equation (5.2) represents the most general and comprehensive condition for the nonbendability of a surface. From (2.15) and (2.16) one recognizes, however, that (5.2) is a complicated partial differential equation for the deformation vector v. Since in solving this differential equation the boundary conditions of the problem must also be fulfilled, one can already see that a general solution of this differential equation will present considerable difficulties.

A handier, although less general, method is using integral formulas. As representative of these, the integral formulas of Herglotz[9, 16] may be mentioned

$$\int_F \frac{\omega}{A}\mathbf{A}\cdot\mathbf{R}\,\mathrm{d}F = \int_C \{(T_g - \tau^*_g)\mathbf{T}\cdot\mathbf{R} - (K_n - \kappa^*_n)\mathbf{N}\cdot\mathbf{R}\}\mathrm{d}C - 2\int_F (H - H^*)\,\mathrm{d}F, \tag{5.3}$$

their "symmetrized" form[9, 16] being

$$\int_F \frac{\omega}{A}(\mathbf{A}\cdot\mathbf{R} + \mathbf{a}^*\cdot\mathbf{r}^*)\mathrm{d}F = \int_C \{(T_g - \tau^*_g)(\mathbf{T}\cdot\mathbf{R} - \mathbf{t}^*\cdot\mathbf{r}^*) - (K_n - \kappa^*_n)(\mathbf{N}\cdot\mathbf{R} - \boldsymbol{\nu}^*\cdot\mathbf{r}^*)\}\mathrm{d}C. \tag{5.4}$$

The line integrals on the right-hand sides are formed along the closed bounding curve C. Introduced as new quantities for the bounding curve of the undeformed surface F have been:

the tangent vector $$\mathbf{T}(C) = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}C}, \quad |\mathbf{T}| = 1, \tag{5.5}$$

the normal vector $$\mathbf{N}(C) = \mathbf{A}\times\mathbf{T}, \quad |\mathbf{N}| = 1, \tag{5.6}$$

the geodesic convolution $$T_g = -\mathbf{N}\cdot\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}C}, \tag{5.7}$$

the normal curvature $$K_n = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}C}\cdot\mathbf{A}, \tag{5.8}$$

the geodesic curvature $$K_g = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}C}\cdot\mathbf{N}. \tag{5.9}$$

The analogous quantities on the bounding curve of the bent surface f^* have been denoted by $\mathbf{t}^*$,

ν^*, τ_g^* and κ_n^*. H and H^* are the mean curvatures of the undeformed and the bent surfaces. The quantity ω is defined by

$$\omega = \det(\omega_{\alpha\beta}) = \omega_{11}\omega_{22} - \omega_{12}\omega_{21}. \tag{5.10}$$

The metric determinant det $(A_{\alpha\beta})$ has been denoted by A.

For the application of the integral formulas (5.3) or (5.4) further statements on the invariant ω/A are required. To begin with,

$$\frac{\omega}{A} \leq 0 \tag{5.11}$$

holds true for F[4, 9] if F and f^* are two isometric surfaces and if, everywhere on F, the expression $K \geq 0$ for the Gaussian curvature applies (**A** and **a*** must be oriented in such a way that H and H^* are not negative). Furthermore, it can be shown that two isometric surfaces F and f^* are congruent if according to [4, 9] the condition

$$\frac{\omega}{A} = 0 \tag{5.12}$$

is fulfilled. At the same time, however, neither the Gaussian curvature K nor, as appropriate, the geodesic curvature Kg of the asymptotic lines may disappear on F in a whole area.

With the eqns (5.4), (5.11), and (5.12) one can now furnish proof of congruity for two surfaces F and f^*. To briefly sketch the mode of procedure: First the boundary integral in (5.4) is made to disappear, so that the surface integral on the left-hand side equals zero. According to (5.11), ω/A, under the given conditions, is not positive. If furthermore the Minkowski supporting functions $\mathbf{A}\cdot\mathbf{R}$ and $\mathbf{a}^*\cdot\mathbf{r}^*$ possess a fixed sign, then it follows that $\omega/A = 0$. In accordance with (5.12), the congruity of both surfaces then follows.

Let the application be illustrated by a simple example. In the case of a closed surface of a convex body (in which $K > 0$ holds everywhere), the boundary integral automatically disappears and of (5.4) only

$$\int_F \frac{\omega}{A}(\mathbf{A}\cdot\mathbf{R} + \mathbf{a}^*\cdot\mathbf{r}^*)\,dF = 0 \tag{5.13}$$

remains. Let the orientation of the surfaces F and f^* be so determined and that **A** and **a*** point inwards and that F and f^* contain zero as the inner point, then all the Minkowski supporting functions are negative. Equation (5.13) can then only be fulfilled if ω/A equals zero, from which, in accordance with (5.12), the congruity of the two convex body surfaces follows. The so-called theorem of convex body surfaces states that a convex body surface is always nonbendable.

Two further remarks must be made in regard to the application of (5.4). First, the restrictions with regard to (5.11) and (5.12) show the limited area of validity, since cylinder, cone, and other ruled surfaces are excluded. Of great importance here, for practical problems, would be a general mathematical verification for an extended application of (5.11), (5.12), and/or (5.4) also to other classes of surfaces, or a mitigation of the conditions pertaining to (5.11) and (5.12). Furthermore one can clearly recognize from the given integral formulas the important influence of the boundary conditions. In order to infer the congruity of unclosed surfaces with positive Gaussian curvature in the way just mentioned, it is necessary that the boundary integral in (5.4) be zero. This would be the case, e.g. if curvature and torsion of the bounding curve did not change. These very rigid demands can, however, in many cases be further mitigated. Some pertinent examples may be found in [4] and [9].

All equations and explanations given hitherto have been applied to "finite" bending. Similar equations can also be given for "infinitesimal" bending. One may, for instance, according to [9], regard the integral formula of Blaschke

$$\int_F (\rho_1{}^1\rho_2{}^2 - \rho_2{}^1\rho_1{}^2)\mathbf{A}\cdot\mathbf{R}\,dF = \frac{1}{2}\int_c \mathbf{R}\cdot\boldsymbol{\rho}\times\boldsymbol{\rho}_{,\beta}T^\beta\,dC \tag{5.14}$$

as an "infinitesimal analogon" to the integral formula of Herglotz (5.3). The torsional vector $\boldsymbol{\rho}$ introduced in (5.14) is defined by

$$\mathbf{w} = \boldsymbol{\rho} \times \mathbf{A}. \tag{5.15}$$

For its partial derivation the representation

$$\boldsymbol{\rho}_{,\beta} = \rho_\beta{}^\alpha \mathbf{A}_\alpha \tag{5.16}$$

was used. It was shown in[9] that

$$\rho_1{}^1\rho_2{}^2 - \rho_2{}^1\rho_1{}^2 = \frac{1}{A}(\omega_{11}\omega_{22} - \omega_{12}\omega_{21}) \triangleq \frac{\omega}{A} \tag{5.17}$$

applies. Accordingly, similar considerations as for (5.3) or (5.4) can be applied to (5.14)—or modified forms of the same. On account of the detailed representations in [4] and [9] this will not be discussed any further. Decisively for the application of (5.14) is again that the boundary integral is caused to disappear and that a similar procedure is used as in (5.13). In this way it is possible, for instance, to directly prove the rigidity of the closed convex body surface, which, as we know from Section 2, does not automatically follow from the previously proved nonbendability. Further proofs of rigidity of unclosed surfaces can again be found in[4] and [9].

More detailed statements as to the nature of the boundary conditions of the surface F in regard to the avoidance of "infinitesimal" bending can be found, for example, in the paper of Kollar[12], who proceeds from the theory of partial differential equations and uses a particular form of the fundamental equation (2.23). If the surface F in the orthogonal Cartesian coordinate system x, y, z is given by

$$z = z(x, y) \tag{5.18}$$

and if one denotes the components of the deflection vector $\mathbf{v}$ in the direction of these coordinates with u, v, w, then it follows immediately from (2.23) that

$$\mathrm{d}x\,\mathrm{d}u + \mathrm{d}y\,\mathrm{d}v + \mathrm{d}z\,\mathrm{d}w = 0. \tag{5.19}$$

After elimination of u and v one finally obtains (see [9])

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 w}{\partial x^2} = 0. \tag{5.20}$$

Concerning this, however, two restrictions are to be noted: Equation (5.20) is valid on the one hand only for "infinitesimal" bending and on the other only for surfaces which can be uniquely projected onto a plane (in[4] and [9] denoted as "caps").

As the statements on the rigidity, nonrigidity, nonbendability, or bendability of an ideal surface only possess a restricted validity for the real structure, one will, in doubtful cases, always endeavour to have an investigation carried out on the real structure concerned; i.e. one must, for purposes of calculation, have recourse to the complete system of equations of the shell theory (statical, geometrical, and physical field equations as well as the pertinent boundary conditions). For cases of practical interest this mode of procedure is regarded by the authors as being the most promising in existence at the present time. We shall now go into details about this procedure, and in the next section we shall give further explanations by means of examples with numerical results.

The equations of a linear theory necessary for a numerical calculation of the complete problem have been substantiated in detail in the previous sections. While it would now seem expedient to proceed from the eqn (2.24), it is thought desirable here to also include the neighbouring area

$$\alpha_{\alpha\beta} \approx 0 \tag{5.21}$$

in the scope of a theory of infinitesimal deformations. This procedure is based on the philosophy that in determining physical problems it is always better not to allow any important parameter to become zero, even where this would make the solution of the problem easier. Rather does this groping forward by means of asymptotic diminishment of the parameter in question lead to new perceptions. With this in mind, that less can be more, the attempt was made by the second-named author to achieve a quantitative demarcation of the theory of inextensional bending (see [2]).

In order to approximate to the limiting value $\alpha_{\alpha\beta} = 0$ as accurately as desired, the following rule of approximation is introduced:

$$\max |A_\alpha| \leq \lambda^{n-r} \max |B_\alpha|. \tag{5.22}$$

This rule says that the absolutely greatest component A_α of a tensor in an area G (apart from singularities) is of a comparable order of magnitude to the amount of quantity B_α, which is representative of the components of another tensor in the same area. With a suitable value (to be chosen arbitrarily)

$$0 < r < 1 \tag{5.23}$$

a range of tolerance can be delimited for any problem as desired. The smaller λ and r are, the better the approximation rule as given in (5.22) will be. Therefore, within the scope of the shell theory underlying these considerations, the problem of inextensional bending is always present if the first difference tensor $\alpha_{\alpha\beta}$ in comparison with the second difference tensor $\omega_{\alpha\beta}$ does not exceed the magnitude λ^2. Thus, in the concrete case of the infinitesimal theory of inextensional bending, the general approximation rule (5.22) will be

$$\max |\alpha_{\alpha\beta}| \leq \lambda^{2-r} \max |\omega_{\alpha\beta}|. \tag{5.24}$$

For example, with $\lambda = 10^{-2}$ and $r = 0.5$, one obtains, in accordance with (5.24)

$$\max |\alpha_{\alpha\beta}| \leq 10^{-3} \max |\omega_{\alpha\beta}| \tag{5.25}$$

a basis for classifying a calculated state of stress and deformation within the limits of the selected uncertainty area as the predominant problem of inextensional bending (see Fig. 1). Making use of the constitutive equations, kinematical relations, and conditions of equilibrium given in the previous sections, the conditional eqn (5.24) indicates whether in a special case, taking into consideration the investigated shell geometry, the existing load distribution, and the given boundary conditions, the problem of inextensional bending is present. This procedure of an asymptotic approximation to the limiting value with $\alpha_{\alpha\beta} = 0$ appears to be reasonable for the vast majority of the practical problems occurring. Numerical results obtained with this procedure will be discussed in the next section.

Should the possibilities of solution hitherto discussed fail, then the method of an experimental model investigation as given by Rothert in [19] can be adopted. This approach can be particularly recommended for certain shells with negative Gaussian curvature because in the case of this type of structures not only the theory of small deformations but also the calculating methods used for shells with positive Gaussian curvature may fail.

6. EXAMPLES AND NUMERICAL RESULTS

A few chosen, characteristic examples will now be given to supplement and illustrate the previous discussions, and here again we shall proceed from "ideal" surfaces. Reference was already made in Section 2 to the nonrigidity of plane surfaces. It generally is valid good that any plane surface which is immovably supported at its edge, although nonrigid, is nonbendable. Precisely this example shows once again very clearly the difference between "infinitesimal" and "finite" bendings.

Like the planes, the cylindrical surfaces also have a vanishing Gaussian curvature. Firstly, it may be shown that any cylindrical surface is bendable, which automatically means that it is nonrigid. This fact leads to the recognition that, in general, cylindrical surfaces will be relatively flexible structures. This situation is altered, however, if one closes the two bases of the cylinder with planes. The lateral area of the cylinder is then rigid (the plane bases remain excepted, since they are, of course, always nonrigid.) Even if one cuts holes in the bases, the rigidity of the lateral area is maintained. These simple examples show very impressively the influence of the boundary conditions on the flexibility of the cylinder. The hindrance to the free deformability of the bounding curve at the bases produces a certain stiffening effect. In shell theory this is taken into account by the requirement that in the case of cylindrical roofs, for instance, in-plane panels at both ends should always be provided.

Considerably stiffer by nature are surfaces with a positive Gaussian curvature. From the standpoint of the engineer this fact was rendered quite conclusively by the previously proved nonbendability and rigidity of the closed convex body surface—the so-called theorem of convex body surfaces. As obvious as this conclusion may be and however often it may have been substantiated by experience with actual structures, a certain amount of caution appears to be indicated. The statement just made was, in fact, only valid for the closed surface. As soon as one cuts a hole in the convex body surface it becomes nonrigid and bendable. Here again the boundary conditions play an important part. For if one takes precautions to see that the bounding curves remain rigid, then the rigidity of the perforated convex body surface is again ensured.

To illustrate the calculations for the quantitative estimation of inextensional bending states—a procedure characterized by the authors in the foregoing section as particularly suitable—a few numerical results will now be given. They are taken from the study of the second-named author[2] and constitute only a fraction of the hitherto evaluated material, while computing programs and evaluation are ascribable to Y. Basar.

By way of example, the mode of procedure in calculating inextensional bending states in a boundary-loaded circular cylindrical shell will be shown. For this purpose the shell shown in Fig. 3 with the radius $r = 10$ m, the height = 15 m, Young's modulus $E = 3 \cdot 10^6$ Mp/m^2, Poisson ratio $\nu = 0$, shell thickness $t = 10$ cm, and $\lambda = 10^{-2}$ was chosen. To illustrate load effects, the displacement of the lower bounding curve will be investigated, in which case the deflection $v_{(3)}$ as well as the tangential displacement $v_{(1)}$ in the direction θ^1 will be predetermined to be

$$v_{(3)} = v_{(n)} \cos n\theta^1, \quad v_{(1)} = b_{11}\sqrt{(a^{11})}v_{(3)}/n \tag{6.1}$$

with

$$v_{(n)} = 0.10\,\text{m} \quad \text{and} \quad n = 1, 2, 3, \ldots \tag{6.2}$$

The choice of the number n as an index shows the association of the nth harmonic developed in Fourier series with a variable of state. The boundary condition chosen in (6.1) corresponds to the proposition $\alpha_{11n} = 0$ for the lower bounding curve.

The qualitative alteration of the displacements of the bounding curve is attained by systematically running through n in accordance with (6.2). In this way one exposes the lower bounding curve to displacements whose degree of variability grows continuously with increasing n. It becomes evident that this qualitatively varying course of the boundary functions considerably influences the stress and deformation state of the structure. The use of various harmonics has the additional advantage that from the known bearing behaviour of each harmonic

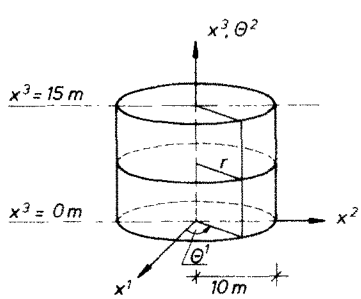

Fig. 3. Geometry of the cylindrical shell.

a conclusion can be drawn in a simple manner as to the stress and deformation state of its combination. Here it should be remembered that as a result of the chosen boundary functions (6.1) or (6.2) the variables of state—as the variables are associated with cos or sin respectively—represent periodic functions. Accordingly, their course is only shown for their maximum values, which correspond to the points $\theta^1 = 0$ or $\theta^2 = \pi/2n$ respectively.

In detail, the calculation produces the following numerical results:

(1) For $n = 1$, the system is subjected to a pure transposition in the direction of the x^1-axis, in which case the amount of the displacement vector is $|v| = 0.10$ m and the variables of deformation and stress resultants vanish identically.

(2) For $n = 2$, the variables of state concerned are shown in Fig. 4. The displacement components $v_{(1)}$ and $v_{(3)}$ run through the entire shell at an almost constant value. The tangential displacements $v_{(2)}$ as compared with $v_{(3)}$ are approximately 10^3 times smaller and therefore negligible. This means to say that the value of deformation of the parallel remains practically unaltered over the whole height. Fig. 5 shows this deformation in a spatial representation. Experimentally this can be demonstrated very simply with the previously-mentioned sheet of paper struck together in the form of a cylinder, which one presses together at opposite points of the lower edge with two fingers. In Fig. 6 the lower bounding curve of the shell is shown before and after deformation. The plotted curves are characterized by the fact that throughout their whole course they show the same arc length. With a predetermined deflection function, therefore, the displacement function can be chosen in a way that the bounding curve does not undergo any change of length when deformation takes place.

The normal forces $n_{(22)}$ and $n_{(12)}$ extend throughout the whole shell and are comparable in their magnitude with the circumferential moments (Fig. 4). The bending moment $m_{(11)}$, which is mainly dependent on $v_{(3)}$, is characterized, like $v_{(3)}$, by a constant course. In this connection attention is drawn to the relatively small bearing reactions with which the predetermined deformation of the bounding curve is produced:

$$n_{(22)} = 1.01 \text{ Mp/m}, \quad n_{(21)} = 0.65 \text{ Mp/m}, \quad q_{(2)} = -0.058 \text{ Mp/m}.$$

From this one can see that the cylindrical shell does not offer any resistance of importance to the longwave boundary displacements.

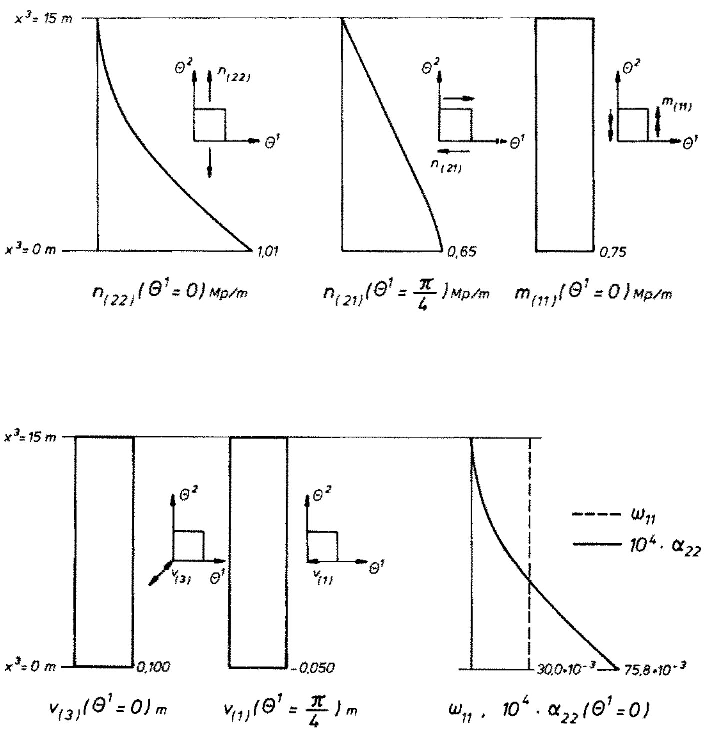

Fig. 4. Numerical results considering $n = 2$.

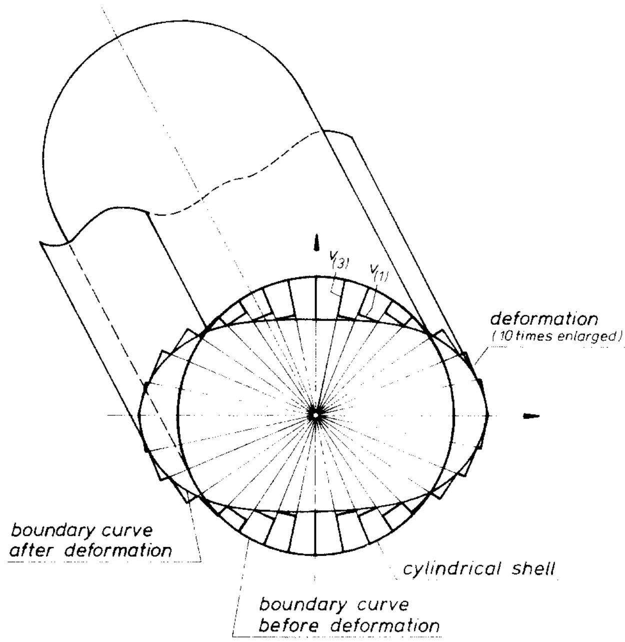

Fig. 5. Deformation of the cylindrical shell for $n = 2$.

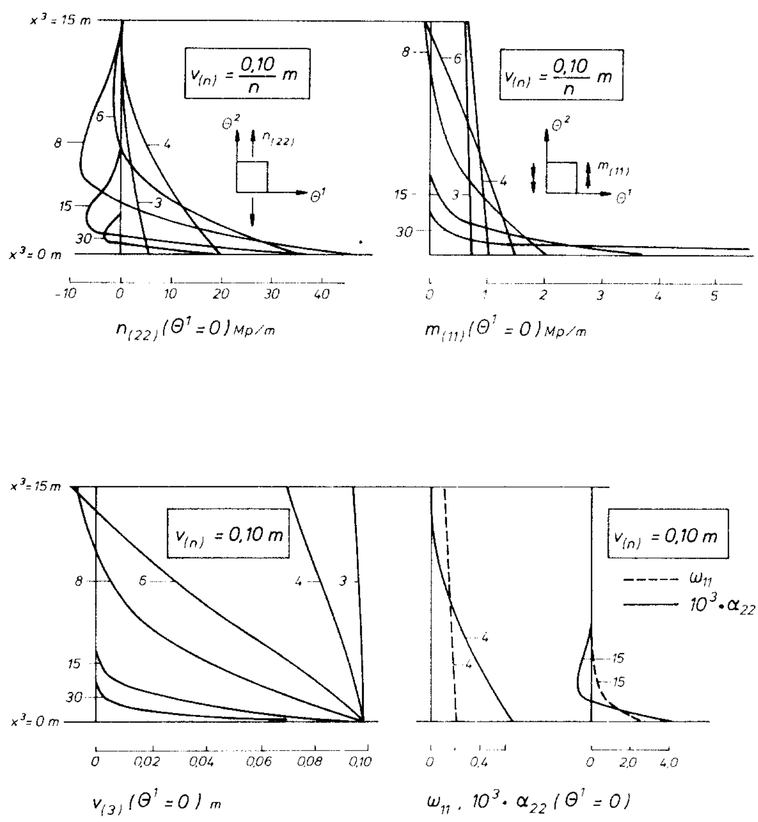

Fig. 6. Numerical results considering harmonics $n > 2$.

In the problem considered the 10^4-times values of the tensor $\alpha_{\alpha\beta}$ are of order of magnitude as the tensor $\omega_{\alpha\beta}$, the most unfavourable value of $\max|\alpha_{22}|/\max|\omega_{11}|$ amounting to approximately 1/4000, whereby the problem, in accordance with the approximation here chosen (5.25), falls within the domain of inextensional bending.

Now we shall look at the results obtained for the higher values of n (Fig. 7). With increasing n, the deflections in the lower bounding surface diminish steadily and increase more and more in the lower bounding curve area, so that with $n = 30$ they practically affect 1/6 of the entire shell. With increasing waviness of the displacement of the bounding curve the predominant deformation of the structure becomes less. With increasing n, the meridian forces $n_{(22)}$ increase so rapidly that their values corresponding to $v_{(n)} = (0.10/n)$ can be drawn in the same diagram. In the case of the circumferential moments $m_{(11)}$ this rapid increase only occurs when higher values of n are reached. From $n = 15$ onwards, the functions mentioned are characterized by a pronounced decay behaviour.

With $n = 3$, the criterion of inextensional bending is slightly infringed only in the vicinity of the lower edge of the boundary curve. This can therefore be regarded as a borderline case. With $n = 4$, however, taking into account the area of uncertainty as determined here, we have a problem of bending theory, as can be seen from a comparison of the strain tensors α_{22} and ω_{11}. Finally, with increasing n, the problem becomes one of an edge disturbance.

The space available does not allow of any further numerical results being shown. In the case of the spherical shell and the hyperboloid of rotation further results can be found in[2]. For the reader of the 1974 German Annual Handbook on Concrete (Betonkalender) it should be mentioned that the problem of a strong increase of the bending moments discussed by Kollar on pages 399 pp. also must be seen in connection with the problem of inextensional bending dealt with in this study.

As the computing program at present in operation is only applicable to rotational shells, the problem of inextensional bending in the case of flat hypar shells has been approached by way of experimental investigations. The results of experimental investigations given by Rothert in[19] confirm very impressively the computational investigations made by Duddeck[3]. In[19] it becomes particularly apparent what constructional measures have to be taken in order to prevent a shell structure from the generally undesirable state of inextensional bending.

7. CONCLUDING REMARKS

This contribution presents a review of the present state of the theory of inextensional bending of the middle surface of shells. It covers the geometrical and the physical problem as well. Main emphasis has been given to various forms of mathematical treatment offering a new approach.

For the borderline case of an ideal surface a mathematically exact definition is possible by means of $\alpha_{\alpha\beta} = 0$. Special investigations can be carried out with the aid of methods of differential geometry. For real structures these propositions are only of limited validity. For many practical problems it is therefore necessary to solve the appertaining system of consistent equations numerically. The approach of a rough approximation as proposed here would therefore seem to hold out particular promise of success. By investigation of the adjacent finite area one obtains numerical results in an "area of uncertainty". Dependent on the limits of toleration chosen, the problem in question can then be classified as predominantly one of bending theory or of inextensional bending. With the aid of the proposed approximation (5.24) one is able to analyse a shell structure quantitatively as well. The numerical results given illustrate the new approach and provide a better understanding of the problem of inextensional bending of shell structures.

If with the procedure as discussed the problem of inextensional or quasi-inextensional bending of the middle surface of a shell can be satisfactorily solved, a number of questions still remains open. For example, are there any constellations of geometry, boundary conditions, or loading conceivable which necessitate an extension of the constitutive equations by resorting to "higher approximations"? What influence does the neglect of the terms of a theory of large deformations mean to the present limitation within the bounds of an infinitesimal theory? What, finally, is the relation of the order of magnitude of these terms which have not been considered in the cases of the behaviour of a nonlinear elastic material?

This, and an almost arbitrarily expandable, hitherto not yet answered catalogue of questions, may make this special case of the shell theory, which because of its mathematical-physical difficulties has received little consideration, to a further profitable field of research.

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